

Dividing goods **and** bads under additive utilities*

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Abstract

When utilities are additive, we uncovered in our previous paper [1] many similarities but also surprising differences in the behavior of the familiar *Competitive rule* (with equal incomes), when we divide (private) *goods* or *bads*. The rule picks in both cases the critical points of the product of utilities (or disutilities) on the efficiency frontier, but there is only one such point if we share goods, while there can be exponentially many in the case of bads.

We extend this analysis to the fair division of *mixed items*: each item can be viewed by some participants as a good and by others as a bad, with corresponding positive or negative marginal utilities. We find that the division of mixed items boils down, normatively as well as computationally, to a variant of an *all goods* problem, or of an *all bads* problem: in particular the task of dividing the non disposable items must be either good news for everyone, or bad news for everyone.

If at least one feasible utility profile is positive, the Competitive rule picks the unique maximum of the product of (positive) utilities. If no feasible utility profile is positive, this rule picks all critical points of the product of *disutilities* on the efficient frontier.

1 Introduction

In our previous paper [1] we consider fair division of (private, divisible) items under linear preferences, represented for convenience by additive utilities. We explain there the appeal of this domain restriction for the practical implementation of division rules vindicated by theoretical analysis. We focus there on the *Competitive Rule* (aka Competitive Equilibrium with Equal Incomes) to divide the items, and contrast its behavior when we divide *goods* (assets, such as family heirlooms, real estate, land, stocks), and when we divide *bads* (chores, workloads, liabilities, noxious substances or facilities). Several normative properties of this rule are identical in both contexts, e. g., *No Envy* and a simple version of Maskin Monotonicity that we call *Independence of Lost Bids*. However the unexpected finding is that several aspects of the rule are very different in the two contexts: dividing *bads* is not a

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mirror image of dividing *goods*. The Competitive Rule picks a unique welfare profile when it divides goods, but for dividing bads it often proposes many (up to exponentially many in the number of agents and bads) allocations with different welfare consequences; in the former case the competitive welfare profile is continuous in the marginal rates of substitution, in the latter case such continuity is not feasible. Also the rule makes every participant benefit from an increase in the goods to divide, a monotonicity property that is out of reach when we divide bads.

Here we generalize this analysis to fair division problems involving (non disposable) *mixed items*, i. e., both goods and bads, or even items about which participants disagree whether they are *good* or *bad*. An inheritance may include good and bad real estate (e. g., heavily mortgaged or not), divorcing couples must allocate jewellery as well as obnoxious pets, workers sharing a multiple jobs relish certain jobs and loath others; and managers facing the division of onerous tasks may deliberately add some desirable items to “sweeten” the deal of the workers.

For a start we show that, upon adapting the standard definition to allow for the co-existence of positive and negative prices and for individual budgets of arbitrary sign, the Competitive Rule is always non empty, and its basic normative properties (*No Envy*, *Independence of Lost Bids*, *Core from Equal Split*¹) are preserved.²

The *status quo ante* situation with nothing to divide delivers in our model zero utility to each participant. If all items are good, any feasible allocation brings (weakly) positive utilities, so that the arrival of the “manna” is good news for everyone; similarly the task of dividing non disposable, undesirable items is a chore for everyone bringing weakly negative utilities to all. With mixed items, some good, some bad, allocations where some participants enjoy positive utility and others negative utility are of course feasible, and some interpretations of fairness pick such divisions (see an example below). Remarkably the Competitive Rule never does: in *any* problem mixing goods and bads, *either* it weakly improves the welfare of *all* agents above the status quo ex ante, *or* it weakly decrease everyone’s welfare below this benchmark. The rule enforces a strong solidarity among agents: the task of dividing any bundle of non disposable items is either unanimously good news or unanimously bad news.

The result We call a problem *positive* if the zero utility profile is strictly below the efficient utility frontier; *negative* if it is strictly above; and *null* if it is on this frontier. In a positive problem the Competitive Rule (CR for short) picks the unique allocation maximizing the Nash product of utilities among positive profiles: it behaves *as if* there are only goods, selects a unique utility profile, and enjoys the same regularity and monotonicity properties as in the all-goods case (within the class of positive problems). In a negative problem the CR picks all the critical points of the product of *disutilities* over the efficient negative profiles: it behaves *as if* there are only bads, in particular it may pick many different utility profiles, and loses its regularity and monotonicity properties. In a null problem the rule implements the null utility profile.

Note that the familiar *Fair Share* utility still sets a lower bound on each agent competitive utility, but these utilities are no longer a useful benchmark as they can be of different signs in the same problem (see an example below).

¹Although in a slightly weaker sense, see Lemma 1.

²Note that existence follows (much) earlier results about competitive equilibrium under satiated (not necessarily linear) preferences such as [2]. Our proof in our special domain is however simpler and constructive.

Some simple examples Here is a two-agent, two-item example where a is a good and b is a bad:

	a	b
u_1	4	-2
u_2	1	-5

There is one unit of each item to share. So a is very good for agent 1 compared to b , while b is very bad relative to a for agent 2.³ Fair Share utilities obtain by giving one half unit of each item to each agent: $U_1^{FS} = 1, U_2^{FS} = -2$. The familiar Egalitarian Rule equalizes the utility gains above this benchmark relative to the maximal feasible gain:

$$\frac{U_1^{ER} - U_1^{FS}}{U_1^{MAX} - U_1^{FS}} = \frac{U_2^{ER} - U_2^{FS}}{U_2^{MAX} - U_2^{FS}} \text{ where } U_1^{MAX} = 4, U_2^{MAX} = 1$$

Combined with Efficiency this gives $U_1^{ER} = 2\frac{2}{7}, U_2^{ER} = -\frac{5}{7}$: see Figure 1. Thus the division task is good news for agent 1 but not for agent 2. By contrast the Competitive Rule focuses on the interval of strictly positive and efficient utility profiles corresponding to the allocations

$$\begin{array}{ccc} & a & b \\ z_1 & x & 1 \\ z_2 & 1-x & 0 \end{array} \text{ where } \frac{1}{2} \leq x \leq 1$$

It picks the midpoint $x = \frac{3}{4}$ with corresponding utilities $U_1^{CR} = 1, U_2^{CR} = \frac{1}{4}$, where agent 1 gets only her Fair Share utility.

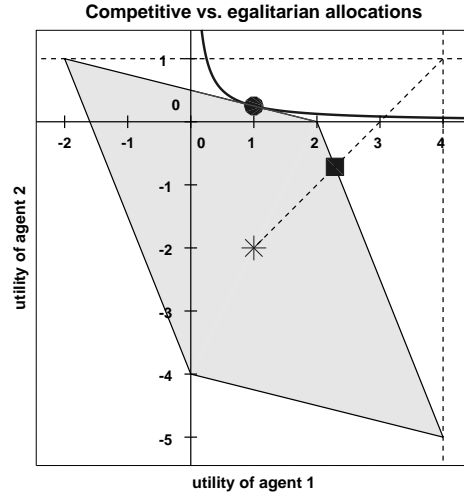


Figure 1: Competitive (circle) and Egalitarian (square) utility profiles for the first example.

Our next example is a negative problem

	a	b
u_1	4	-5
u_2	1	-5

³Recall interpersonal comparisons of utilities are ruled out, only the underlying preferences matter.

where the efficient allocations with strictly negative utility profiles cover the interval

$$\begin{array}{cc} a & b \\ z_1 & 0 \quad x \\ z_2 & 1 \quad 1-x \end{array} \quad \text{where } \frac{4}{5} \leq x \leq 1$$

so the competitive allocation is at $x = \frac{9}{10}$ with utilities $U_1^{CR} = U_2^{CR} = -\frac{1}{2}$, where again agent 1 gets only her Fair Share utility. The ER utilities are $U_1^{ER} = \frac{2}{5}, U_2^{ER} = -\frac{7}{5}$.

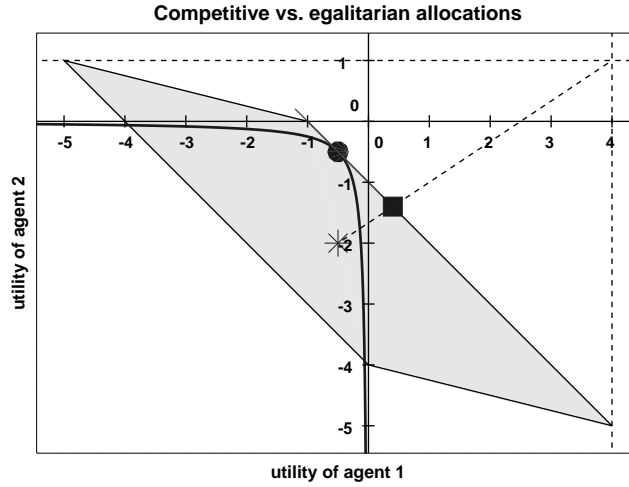


Figure 2: Competitive (circle) and Egalitarian (square) utility profiles for the second example.

In Section 5 we compute the competitive division in a sequence of problems with two agents, two bads, and a third item starting as a good and becoming increasingly bad. The initial problem is positive, then becomes negative; the number of competitive allocations takes all values from 1 to 4.

We stress another key difference between the Competitive and Egalitarian Rules, implied by Independence of Lost Bids. If an object a is a good for some agents and a bad for others, efficiency implies that only the former agents eat it. The CR ignores the detailed disutilities of the latter agents: nothing changes if we set those disutilities to zero, so that a becomes a good. This means that we need only to consider problems where items are either (at least weakly) good for everyone, or (at least weakly) bad for everyone. Obviously this simplification does not apply to the ER.

2 The model

The set of agents is N with cardinality n , that of objects is A . A problem is $\mathcal{P} = (N, A, u \in \mathbb{R}^{N \times A})$ where the utility matrix u has no null column.

With the notation $z_M = \sum_{i \in M} z_i$, and e^B for the vector in \mathbb{R}^B with $e_b^B = 1$ for all b , we define a feasible allocation as $z \in \mathbb{R}_+^{N \times A}$ such that $z_N = e^A$. Let $\mathcal{F}(\mathcal{P})$ be the set of feasible

allocations, and $\Phi(\mathcal{P})$ the corresponding set of utility profiles. We always omit \mathcal{P} if it is clear from the context.

We call a feasible utility profile *efficient* if it is not Pareto dominated⁴; also a feasible allocation is efficient if it implements an efficient utility profile.

The following two partitions, of N and A respectively, are critical:

$$N_+ = \{i | \exists a : u_{ia} > 0\}, N_- = \{i | \forall a : u_{ia} \leq 0\}$$

$$A_+ = \{a | \exists i : u_{ia} > 0\}, A_- = \{a | \forall i : u_{ia} < 0\}, A_0 = \{a | \max_i u_{ia} = 0\}$$

When no confusion may arise, we call an object in A_+ *good*, one in A_- *bad*, and one in A_0 *neutral*.

Definition 1: For any problem \mathcal{P} a competitive division is a triple $(z \in \mathcal{F}, p \in \mathbb{R}^A, \beta \in \{-1, 0, +1\})$ where z is the competitive allocation, p is the price and β the budget. The allocation z_i maximizes i 's utility in the budget set $B(p, \beta) = \{y_i \in \mathbb{R}_+^A | p \cdot y_i \leq \beta\}$:

$$z_i \in d_i(p, \beta) = \arg \max_{B(p, \beta)} \{u_i \cdot y_i\} \quad (1)$$

Moreover z_i minimizes i 's wealth in her demand set

$$z_i \in \arg \min_{d_i(p, \beta)} \{p \cdot y_i\} \quad (2)$$

The Competitive Rule selects at each problem \mathcal{P} the set $CR(\mathcal{P})$ of all competitive allocations.

In addition to the usual demand property (1), we insist that an agent spends as little as possible for her competitive allocation. This requirement appears already in [2]: in its absence some satiated agents in N_- may inefficiently eat some objects useless to themselves

but useful to others. For instance in the two agents-two item problem $u = \begin{pmatrix} 6 & 2 \\ 0 & -1 \end{pmatrix}$ the

inefficient allocation $z = \begin{pmatrix} 1/3 & 1 \\ 2/3 & 0 \end{pmatrix}$ meets (1) for the prices $p = (\frac{3}{2}, \frac{1}{2})$ and budget $\beta = 1$.

However $z_2 = (0, 0)$ guarantees the same (zero) utility to agent 2 and costs zero, so it fails (2). The only competitive division according to the Definition is $z = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ for $p = (\frac{1}{2}, \frac{1}{2})$.

Check that in a competitive division we have

$$p_a > 0 \text{ for } a \in A_+ ; p_b < 0 \text{ for } b \in A_- ; p_a = 0 \text{ for } a \in A_0 \quad (3)$$

If the first statement fails an agent who likes a would demand an infinite amount; if the second fails no one would demand b . If the third fails with $p_a > 0$ the only agents who demand a have $u_{ia} = 0$, so that eating some a violates (2); if it fails with $p_a < 0$ an agent such that $u_{ia} = 0$ gets an arbitrarily cheap demand by asking large amounts of a , so (2) fails again.

Here is another consequence of (2), the importance of which is illustrated by the above example:

$$\forall a \in A_+ : z_{ia} > 0 \implies u_{ia} > 0 \quad (4)$$

⁴That is $U \in \Phi(\mathcal{P})$, and if $U \leq U'$ and $U' \in \Phi(\mathcal{P})$, then $U' = U$.

Indeed if i eats some $a \in A_+$ and $u_{ia} = 0$, she gets a cheaper competitive demand by ignoring a ; and if $u_{ia} < 0$ her allocation is not competitive (recall $p_a > 0$).

We recall three standard normative properties of an allocation $z \in \mathcal{F}(\mathcal{P})$. It is *Non Envious* iff $u_i \cdot z_i \geq u_i \cdot z_j$ for all i, j ; it guarantees the *Fair Share* utilities iff $u_i \cdot z_i \geq u_i \cdot (\frac{1}{n}e^A)$ for all i . It is in the *Weak Core from Equal Split* if for all $S \subseteq N$ and all $y \in \mathbb{R}_+^{S \times A}$ such that $y_S = \frac{|S|}{n}e^A$ there is at least one $i \in S$ such that $u_i \cdot z_i \geq u_i \cdot y_i$. When we divide goods competitive allocations meet these three properties, even in the much larger Arrow-Debreu preference domain.

Lemma 1 *A competitive allocation is efficient; it meets No Envy, guarantees the Fair Share utilities, and is in the Weak Core from Equal Split.*

Proof.

Efficiency. The classic argument by contradiction can be adapted here. Let z be a competitive allocation Pareto inferior to the feasible allocation y . Some agent i^* strictly prefers y_{i^*} to z_{i^*} which implies $p \cdot z_{i^*} < p \cdot y_{i^*}$ by (1). So if we show $p \cdot z_i \leq p \cdot y_i$ for all i , we contradict $z_N = y_N$ by summing up these inequalities. Note that we can assume that y itself is efficient which will be useful below.

First we have $p \cdot z_i = \beta$ for $i \in N_+$, or i could buy more of an object in A_+ he likes; moreover i prefers strictly z_i to any y_i such that $p \cdot y_i < \beta$, and weakly if $p \cdot y_i \leq \beta$. So $p \cdot z_i \leq p \cdot y_i$ for all $i \in N_+$. It remains to show $p \cdot z_j \leq p \cdot y_j$ for all $j \in N_-$.

We distinguish two cases. If $\beta = 0, +1$ we have $u_j \cdot z_j = 0$ (the best feasible utility for j) and $u_j \cdot y_j = 0$ as well. At z agent j can only consume objects in A_0 : by (4) j eats no item in A_+ and eating in A_- strictly lowers her utility; agent j eats no object in A_+ at y either by efficiency of y , and by $u_j \cdot y_j = 0$ she eats nothing in A_- as well. Objects in A_0 are free ((3)) so $p \cdot z_j = p \cdot y_j = 0$.

Now if $\beta = -1$ an agent j in N_- must eat some objects in A_- that he dislikes hence his competitive demand z_j has $p \cdot z_j = \beta$ and as above he strictly prefers z_j to y_j if $p \cdot y_j < \beta$: so $p \cdot z_j \leq p \cdot y_j$ as desired.

Other properties. No Envy is clear and it implies Fair Share by additivity of utilities. We use again the standard argument to check the Weak Core property. Assume coalition $S \subset N$ has an objection y to the competitive division (z, p, β) where everybody in S strictly benefits. So $y \in \mathbb{R}_+^{S \times A}$ and $u_i \cdot z_i < u_i \cdot y_i$ for all $i \in S$. If $\beta = 0, +1$, this inequality is impossible for $i \in N_-$ because $u_i \cdot z_i = 0$ (see above), so $S \subseteq N_+$. Then we sum over S the inequalities $p \cdot z_i < p \cdot y_i$ to get

$$|S|\beta = p \cdot z_S < p \cdot y_S = p \cdot \frac{|S|}{n}e^A$$

which contradicts $p \cdot e^A = p \cdot z_N \leq n\beta$.

If $\beta = -1$ we have $p \cdot z_i = \beta$ for all i , which simplifies the argument. ■

Remark 1: For positive problems a competitive allocation may fail the standard Core from Equal Split property, where coalition S blocks allocation z if it can use its endowment $\frac{|S|}{n}e^A$ to make everyone in S weakly better off and at least one agent strictly more. This is because “equal split” gives resources to the agents in N_- that they have no use for. Say three agents share one unit of item a with $u_1 = u_2 = 1, u_3 = -1$. Here CR splits a between agents 1 and 2, which coalition $\{1, 3\}$ blocks by giving $\frac{2}{3}$ of a to agent 1.

Remark 2: It is easy to check that CR meets *Independence of Lost Bids*, the translation of Maskin Monotonicity under linear preferences: see the precise definition in [1]. Just as

in Proposition 2 of that paper, CR is characterized by, essentially, combining this property with Efficiency.

3 The result

The key to classify our problems when N and A are given is the relative position of the set of feasible utility profiles Φ and the cone $\Gamma(N) = \mathbb{R}_+^{N_+} \times \{0\}^{N_-}$, which can only be of three types. Write the relative interior of Γ as $\Gamma^*(N) = \mathbb{R}_{++}^{N_+} \times \{0\}^{N_-}$.

Definition 2 We call the problem $\mathcal{P} = (N, A, u)$

positive if $\Phi(\mathcal{P}) \cap \Gamma^*(N) \neq \emptyset$;

negative if $\Phi(\mathcal{P}) \cap \Gamma(N) = \emptyset$;

null if $\Phi(\mathcal{P}) \cap \Gamma(N) = \{0\}$.

We fix \mathcal{P} and check that these three cases are exhaustive and mutually exclusive. This amounts to show that $\Phi \cap \Gamma^* = \emptyset$ and $\Phi \cap \partial\Gamma \neq \emptyset$ together imply $\Phi \cap \Gamma = \{0\}$. Pick U non zero in $\Phi \cap \partial\Gamma$ and derive a contradiction. Let $U_i > 0$ for the agents in $P \subset N_+$ and $U_j = 0$ for those in $Q = N_+ \setminus P$. If some $i \in Q$ eats some a he likes ($u_{ia} > 0$), he must also eat some b he dislikes ($u_{ib} < 0$): then let someone in P take a small amount of b from i and we get a new $U' \in \Phi \cap \partial\Gamma$ where P' is larger than P . If no j in Q eats any a she likes (so she does not eat any she dislikes either), we pick any $i \in Q$ and an item a she likes; a must be eaten at U exclusively by some agents in $P \cup N_-$; if some $j \in P$ eats a we let j give a small amount of a to i and we have found $U' \in \Phi \cap \partial\Gamma$ with a larger P' ; if some k in N_- eats some a we have $u_{ka} = 0$ so again k can give his share of a to i and P increases. Repeating this construction until $P = N_+$ we reach $U \in \Phi \cap \Gamma^*$, the desired contradiction.

Given a smooth function f and a closed convex C we say that $x \in C$ is a critical point of f in C if the supporting hyperplane of the upper contour of f at x supports C as well:

$$\forall y \in C : \partial f(x) \cdot y \leq \partial f(x) \cdot x \text{ or } \forall y \in C : \partial f(x) \cdot y \geq \partial f(x) \cdot x$$

This holds in particular if x maximizes or minimizes f in C .

In the statement we write Φ^{eff} for the set of efficient utility profiles, and \mathbb{R}_-^N for the interior of \mathbb{R}_-^N .

Theorem Fix a problem $\mathcal{P} = (N, A, u)$.

- i) The problem \mathcal{P} has a competitive division with a positive budget if and only if it is positive. In this case an allocation is competitive iff it maximizes the product $\prod_{N_+} U_i$ over $\Phi \cap \Gamma^*$; thus the corresponding utility profile is unique, positive in N_+ and zero in N_- .
- ii) The problem \mathcal{P} has a competitive division with a negative budget if and only if it is negative. In this case an allocation is competitive iff it is a critical point of the product $\prod_N |U_i|$ in Φ that belongs to $\Phi^{eff} \cap \mathbb{R}_-^N$. All utilities are negative in any competitive allocation.
- iii) The problem \mathcal{P} has a competitive allocation with a zero budget if and only if it is null. In this case an allocation is competitive iff its utility profile is zero.

Note that the Theorem implies in particular that $CR(\mathcal{P})$ is non empty for all \mathcal{P} .

4 Proof

First we give a closed form description of the competitive demands per Definition 1, i. e., the solutions of (1) plus (2).

Lemma 2 Fix \mathcal{P} a budget $\beta \in \{-1, 0, +1\}$ and a price p such that $p_a > 0$ in A_+ , $p_b < 0$ in A_- , and $p_a = 0$ in A_0 .

i) if $i \in N_-$ and $\beta = 0, +1$, the allocation $z_i \in \mathbb{R}_+^A$ is competitive iff $u_i \cdot z_i = 0$ (for instance $z_i = 0$)

ii) if $i \in N_-$ and $\beta = -1$, $z_i \in \mathbb{R}_+^A$ is competitive iff $p \cdot z_i = -1$, $z_{ia} = 0$ on A_+ , $z_{ia} > 0$ on A_0 only if $u_{ia} = 0$, and

$$\{b \in A_- \text{ and } z_{ib} > 0\} \implies \frac{|u_{ib}|}{|p_b|} \leq \frac{|u_{ib'}|}{|p_{b'}|} \text{ for all } b' \in A_- \quad (5)$$

iii) if $i \in N_+$ the problem (1) has a bounded solution iff

$$\frac{u_{ia}}{p_a} \leq \frac{|u_{ib}|}{|p_b|} \text{ for all } i \in N_+, a \in A_+, b \in A_- \quad (6)$$

Then the allocation z_i is competitive iff $p \cdot z_i = \beta$, and z_i meets (5) and the two following properties:

$$\{a \in A_+ \text{ and } z_{ia} > 0\} \implies \frac{u_{ia}}{p_a} \geq \frac{u_{ia'}}{p_{a'}} \text{ for all } a' \in A_+ \quad (7)$$

$$\{a \in A_+, b \in A_-, \text{ and } z_{ia} > 0, z_{ib} > 0\} \implies \frac{u_{ia}}{p_a} = \frac{|u_{ib}|}{|p_b|} \quad (8)$$

Statement i) is clear upon noticing that eating some object in A_0 is free so that (2) holds. For ii) observe that to meet the budget constraint i must be buying some $b \in A_-$; if i buys some $a \in A_+$ she can increase her utility by buying less of b and of a ; and (2) still holds because objects in A_0 are free. Property (5) simply says that she buys objects in A_- with the smallest disutility per unit of (fiat) money.

For statement iii) pick agent i in N_+ and note that a budget balanced purchase of both objects $a \in A_+$ and $b \in A_-$ increases strictly i 's utility iff $\frac{u_{ia}}{p_a} > \frac{|u_{ib}|}{|p_b|}$, in which case (1) has no bounded solution. As already noted in the proof of Lemma 1, i can buy some object he likes in A_+ with any slack budget, therefore $p \cdot z_i = \beta$, which implies (2). Properties (5) follow as for agents in N_- and (7) is similar. Finally if i eats both $a \in A_+$ and $b \in A_-$, inequality $\frac{u_{ia}}{p_a} < \frac{|u_{ib}|}{|p_b|}$ implies that a budget neutral reduction of z_{ia} and z_{ib} increases U_i : thus we need (8) as well. ■

Proof of the Theorem

Step 1: Statement i). Let $\mathcal{P} = (N, A, u)$ be a positive problem. We show that the maximization of the Nash product on $\Phi \cap \Gamma$ finds a Competitive allocation with a positive budget.

As $\Phi \cap \Gamma$ is compact and convex, there is a unique U^* maximizing in $\Phi \cap \Gamma$ the product $\prod_{N_+} U_i$; clearly $U_i^* > 0$ for all $i \in N_+$. Let z^* be an allocation implementing U^* . By efficiency the items in A_0 are only eaten by agents in N_+ and/or N_- , who do not care about them ($u_{ia} = 0$); this implies

$$\forall i \in N_+ : u_{iA_0} \cdot z_{iA_0}^* = 0 \quad (9)$$

(with the notation $u_{iB} \cdot z_{iB} = \sum_B u_{ib} z_{ib}$).

By efficiency the items in A_+ are eaten in full by N_+ (property (4)); ditto for the items in A_- because for such items and any $i \in N_-$, we have $u_{ia} < 0$ and $U_i^* = 0$. We define prices as follows:

$$\forall a \in A_+ : p_a = \max_{N_+} \frac{u_{ia}}{U_i^*} > 0; \quad \forall b \in A_- : p_b = -\min_{N_+} \frac{|u_{ib}|}{U_i^*} < 0 \quad (10)$$

and $p_a = 0$ on A_0 .

Pick any $a \in A_+$ and $i \in N_+$ eating a (such i exists by the argument above): then $u_{ia} > 0$ by efficiency, so the FOC of the maximization program implies $\frac{\partial}{\partial z_{ja}} \ln(u_j \cdot z_j^*) \leq \frac{\partial}{\partial z_{ia}} \ln(u_i \cdot z_i^*) \iff \frac{u_{ia}}{U_i^*} \geq \frac{u_{ja}}{U_j^*}$ for all $j \in N_+$. This is property (7). Checking (5) is similar: assume $i \in N_+$ eats $b \in A_-$ and recall $u_{ib} < 0$, so the FOCs give $\frac{|u_{ib}|}{U_i^*} \leq \frac{|u_{jb}|}{U_j^*}$ for all $j \in N_+$. Now we fix $i \in N_+$ and apply (7), (5):

$$\{a \in A_+ \text{ and } z_{ia}^* > 0\} \implies U_i^* = \frac{u_{ia}}{p_a} ; \{b \in A_- \text{ and } z_{ib}^* > 0\} \implies U_i^* = \frac{|u_{ib}|}{|p_b|} \quad (11)$$

Summing up numerator and denominator over the support of z_i^* (the items he eats) and invoking (9) as well as $p_a = 0$ on A_0 , we get

$$U_i^* = \frac{\sum_{A_+} u_{ia} z_{ia}^* - \sum_{A_-} |u_{ib}| z_{ib}^*}{\sum_A p_a z_{ia}^*} = \frac{u_i \cdot z_i^*}{p \cdot z_i^*} \implies p \cdot z_i^* = 1$$

as required by Lemma 2.

If A_- is non empty, there is at least one agent $i \in N_+$ eating $a \in A_+$ and $b \in A_-$. For any such i property (11) gives $\frac{u_{ia}}{p_a} = U_i^* = \frac{|u_{ib}|}{|p_b|}$, which proves (8), and

$$\text{for all } a' \in A_+, b' \in A_- : \frac{u_{ia'}}{p_{a'}} \leq U_i^* \leq \frac{|u_{ib'}|}{|p_{b'}|}$$

implying (6).

Step 2: Statement i). Suppose the problem $\mathcal{P} = (N, A, u)$ has a competitive division $(z, p, +1)$. We show that \mathcal{P} is positive and z maximizes the Nash product as in Step 1.

Because $z_i = 0$ is in the budget set, all agents in N_- must get zero utility. If they consume anything, it must be an object in A_0 by assumption (4). Each i in N_+ can buy some amount of any object, so at z her utility is positive: $U_i = u_i \cdot z_i > 0$. Therefore \mathcal{P} is positive.

Fix $i \in N_+$ and recall objects in A_0 , if any, have zero price (assumption (3)). Thus if i eats some $a \in A_0$ we have $u_{ia} = 0$, otherwise i benefits by simply stop eating a . This gives $u_{iA_0} \cdot z_{iA_0} = 0$ (as in (9)), and $p_{A_0} \cdot z_{iA_0} = 0$ as well.

We also know that p is positive on A_+ and negative on A_- , and that $p \cdot z_i = 1$ for all $i \in N_+$ (by efficiency of z). Write the sets of objects i eats in $A_+ \cup A_-$ as $A_+(i) \cup A_-(i)$: $A_+(i)$ is non empty because $U_i > 0$ ($A_-(i)$ can be empty). By Lemma 2 $\frac{u_{ia}}{p_a}$ is constant on $A_+(i)$, and equal to $\frac{|u_{ib}|}{|p_b|}$ on $A_-(i)$ if the latter is non empty. Thus this common ratio is also

$$\frac{u_{iA_+(i)} \cdot z_{iA_+(i)} + u_{iA_-(i)} \cdot z_{iA_-(i)}}{p_{A_+(i)} \cdot z_{iA_+(i)} + p_{A_-(i)} \cdot z_{iA_-(i)}} = \frac{U_i}{p \cdot z_i} = U_i \quad (12)$$

(where the first equality uses $u_{iA_0} \cdot z_{iA_0} = p_{A_0} \cdot z_{iA_0} = 0$). Therefore

$$\frac{u_{ia}}{U_i} = p_a \text{ for all } a \in A_+(i); \frac{|u_{ib}|}{U_i} = |p_b| \text{ for all } b \in A_-(i)$$

Then (7) implies $\frac{u_{ia}}{p_a} \leq U_i$ for all $a \in A_+$ while (6) implies $U_i \leq \frac{|u_{ib}|}{|p_b|}$ for all $b \in A_-$. These two facts together give for all $i \in N_+$:

$$\frac{u_{ia}}{U_i} \leq p_a \text{ for all } a \in A_+ \text{ and } \frac{|u_{ib}|}{U_i} \geq |p_b| \text{ for all } b \in A_- \quad (13)$$

From this we derive that U maximizes $\Pi_{N_+} U_i$ in $\Phi \cap \Gamma$, or equivalently that it is critical for the product of utilities in $\Phi \cap \Gamma$: the restriction to N_+ of any feasible utility profile is below the hyperplane supporting $\Pi_{N_+} U_i$ at (the restriction of) U :

$$\text{for all } U' \in \Phi : \sum_{i \in N_+} \frac{U'_i}{U_i} \leq n$$

Pick $z' \in \mathcal{F}$ implementing U' and use (13) to compute (recalling that p is zero on A_0)

$$\sum_{i \in N_+} \frac{u_i \cdot z'_i}{U_i} = \sum_{a \in A} \sum_{i \in N_+} \frac{u_{ia} z'_{ia}}{U_i} \leq \sum_{a \in A_+} \sum_{i \in N_+} p_a z'_{ia} - \sum_{a \in A_-} \sum_{i \in N_+} |p_b| z'_{ib} = \sum_{a \in A} p_a = n$$

Step 3: Statement ii). Let $\mathcal{P} = (N, A, u)$ be a negative problem. We show there exists a critical point U^* of the product $\Pi_N |U_i|$ in Φ that belongs to $\Phi^{eff} \cap \mathbb{R}^N_-$. The profile U^* we construct maximizes this product on $\Phi^{eff} \cap \mathbb{R}^N_-$, and any allocation z^* implementing it is competitive.

Substep 3.1: If $U^* \in \Phi^{eff} \cap \mathbb{R}^N_-$ is a critical point of $\Pi_N |U_i|$ in Φ , then any z^* implementing U^* is competitive.

We pick an allocation z^* implementing U^* and mimick the argument in Step 1 above. While objects in A_+ are still eaten exclusively by N_+ , those in A_- are eaten by anyone (and everyone eats at least one object in A_-). If $a \in A_+$ and $z^*_{ia} > 0$ for $i \in N_+$, a transfer of some a from i to $j \in N_+$ leaves the allocation on the same side of H as Φ , i. e., below: this implies $\frac{u_{ia}}{|U_i^*|} \geq \frac{u_{ja}}{|U_j^*|}$; if $z^*_{ib} > 0$ for $b \in A_-$ (and $i \in N$), we consider similarly a transfer of some b from i to j to get $\frac{|u_{ib}|}{|U_i^*|} \leq \frac{|u_{jb}|}{|U_j^*|}$. Then we define p in $A_+ \cup A_-$ as in (10), upon replacing U^* by $|U^*|$ and minimizing over all N instead of just N_+ when defining p in A_- . The analog of (11) follows, with the same changes, and the same computation yields $p \cdot z^*_i = -1$, this time for all i .

Setting $p_a = 0$ on A_0 , we now use Lemma 2 to check as in Step 1 that z^*_i is i 's competitive for p and $\beta = -1$.

Substep 3.2: We show that the profile U^* maximizing $\Pi_N |U_i|$ in $\Phi^{eff} \cap \mathbb{R}^N_-$ is a critical point of this product in Φ (and is in \mathbb{R}^N_-).

We have $u_i \cdot e^A < 0$ for every $i \in N$, else the allocation $z_i = e^A$ yields utilities in Γ . Consider the set F of utility profiles dominated by Φ : $F = \{U \in \mathbb{R}^N_- | \exists U' \in \Phi : U' \leq U\}$. This set is closed and convex, and contains all points in \mathbb{R}^N_- that are sufficiently far from the origin: any $U \in \mathbb{R}^N_-$ such that $U_N \leq \min_i u_i \cdot e^A$ is dominated by the utility profile of $z : z_i = \frac{|U_i|}{|U_N|} e^A, i \in N$.

Fix $\tau \geq 0$ and consider the upper contour of the Nash product at τ : $K(\tau) = \{U \in \mathbb{R}^N_- | \Pi_N |U_i| \geq \tau\}$. For sufficiently large τ the closed convex set $K(\tau)$ is contained in F . Let τ_0 be the minimal τ with this property. Negativity of \mathcal{P} implies that F is bounded away from 0 so that τ_0 is strictly positive. By definition of τ_0 the set $K(\tau_0)$ touches the boundary of F at some U^* with strictly negative coordinates. Let H be a hyperplane supporting F at U^* . By the construction, this hyperplane also supports $K(\tau_0)$, therefore U^* is a critical point of the Nash product on F : that is, U^* maximizes $\sum_{i \in N} \frac{U_i}{|U_i^*|}$ over all $U \in F$. So U^* belongs to Pareto frontier of F , which is clearly contained in the Pareto frontier of Φ . Thus U^* is a critical point of the Nash product on Φ and belongs to $\Phi^{eff} \cap \mathbb{R}^N_-$. Any U in the interior of

$K(\tau_0)$ is clearly dominated by some U' in $K(\tau_0) \subset F$, hence by some $U'' \in \Phi \cap \mathbb{R}_-^N$: so U^* maximizes the Nash product on $\Phi^{eff} \cap \mathbb{R}_-^N$.

Remark 3: Note that the supporting hyperplane H to Φ at U^* is unique because it is also a supporting hyperplane to $K(\tau_0)$ that is unique. This means that U^* belongs to a face of a polytope Φ of maximal dimension.

Step 4: Statement ii). Suppose the problem $\mathcal{P} = (N, A, u)$ has a competitive division $(z, p, -1)$. We show that \mathcal{P} is negative and the corresponding utility profile U is a critical point of the product $\Pi_N |U_i|$ in Φ that belongs to $\Phi^{eff} \cap \mathbb{R}_-^N$.

The utility of any agent in N_- at z is negative: goods in A_0 are free ((3)), he does not eat any object in A_+ ((4)), and his budget is negative. Applying Lemma 2 to an agent in N_+ we see, as in Step 2, that the ratio $\frac{u_{ia}}{p_a}$ is constant on $A_+(i)$, and equal to $\frac{|u_{ib}|}{|p_b|}$ on $A_-(i)$. The same computation (12) gives $\frac{u_{ia}}{p_a} = \frac{|u_{ib}|}{|p_b|} = \frac{U_i}{p \cdot z_i} = -U_i$, so $U_i < 0$ in N_+ as well. By Lemma 1 the negative profile U is efficient, so Φ cannot intersect \mathbb{R}_+^n and \mathcal{P} is negative.

We derive now the criticality of U for $\Pi_N |U_i|$ much like we did in Step 2. The difference is that now *everyone* eats some object in A_- , and i in N_+ may or may not eat some object in A_+ (but i in N_- still doesn't).

From $|U_i| = \frac{u_{ia}}{p_a}$ on $A_+(i)$, $= \frac{|u_{ib}|}{|p_b|}$ on $A_-(i)$ we get (2) (for all i , and with $|U_i|$ instead of U_i). Because any $i \in N$ must eat some b^* in A_- , properties (5) and (2) yield $|U_i| = \frac{|u_{ib^*}|}{|p_{b^*}|} \leq \frac{|u_{ib}|}{|p_b|}$ for all $b \in A_-$. Then (6) gives $\frac{u_{ia}}{p_a} \leq |U_i|$ for all $a \in A_+$, and this implies the analog of property (13): for all $i \in N$

$$\frac{u_{ia}}{|U_i|} \leq p_a \text{ for all } a \in A_+ \text{ and } \frac{|u_{ib}|}{|U_i|} \geq |p_b| \text{ for all } b \in A_- \quad (14)$$

The criticality of $U \in \Phi^{eff} \cap \mathbb{R}_-^N$ for $\Pi_N |U_i|$ in Φ means now that all feasible utility profiles are below the hyperplane supporting $\Pi_N |U_i|$ at U

$$\text{for all } U' \in \Phi : \sum_{i \in N} \frac{U'_i}{|U_i|} \leq -n$$

(but this time U does not maximize $\Pi_N |U_i|$ on all of Φ). The derivation of this inequality from (14) proceeds exactly as in Step 2.

Step 5: Statement iii). Let $\mathcal{P} = (N, A, u)$ be a null problem. We show there exists a price p such that $(z, p, 0)$ is competitive iff $u_i \cdot z_i = 0$ for all i .

As $\Phi \cap \Gamma = \{0\}$ we can separate the projection of Φ on \mathbb{R}^{N_+} from $\mathbb{R}_+^{N_+}$: there exists $\lambda \in \mathbb{R}_+^{N_+} \setminus \{0\}$ such that $\sum_{i \in N_+} \lambda_i U_i \leq 0$ for all $U \in \Phi$. If $\lambda_i = 0$ for some $i \in N_+$ we pick $j \in N_+$ such that $\lambda_j > 0$ and the allocation where j eats an object she likes and i eats all the rest yields a contradiction. Thus λ is strictly positive.

Pick any $z^* \in \mathcal{F}$ implementing $U = 0$: it is efficient therefore $u_{iA_0} \cdot z_{iA_0}^* = 0$. We have

$$\begin{aligned} z^* &\in \arg \max \left\{ \sum_{N_+} \lambda_i (u_i \cdot z_i) \mid z \in \mathcal{F} \right\} = \\ &= \arg \max \left\{ \sum_{a \in A_+} \left(\sum_{N_+} \lambda_i u_{ia} z_{ia} \right) - \sum_{b \in A_-} \left(\sum_{N_+} \lambda_i |u_{ib}| z_{ib} \right) \mid z \in \mathcal{F} \right\} \end{aligned}$$

We define the price p as

$$\forall a \in A_+ : p_a = \max_{N_+} \lambda_i u_{ia}; \quad \forall b \in A_- : p_b = - \min_{N_+} \lambda_i |u_{ib}|$$

and as usual $p = 0$ in A_0 . Clearly p is positive on A_+ and negative on A_- . On the support of z_i^* we have

$$\forall a \in A_+ \cup A_- : z_{ia}^* > 0 \implies p_a = \lambda_i u_{ia} \quad (15)$$

implying $p \cdot z_i^* = \lambda_i (u_i \cdot z_i^*) = 0$. The definition of p implies $p_a \geq \lambda_i u_{ia}$ and $|p_b| \leq \lambda_i |u_{ib}|$ for all $i \in N_+, a \in A_+, b \in A_-$. This implies (6) at once; together with (15) it gives (7); the proof of (5) and (8) is similar.

Step 6: Statement iii). Suppose the problem $\mathcal{P} = (N, A, u)$ has a competitive division $(z, p, 0)$. We show that \mathcal{P} is null.

Let U be the utility profile of z . Clearly $U_i = 0$ for $i \in N_-$. Fix $i \in N_+$ such that $z_i \neq 0$: because $p \cdot z_i = 0$ (Lemma 2) i must eat at least one object in A_+ and one in A_- . By Lemma 2 again we have $\frac{u_{ia}}{p_a} = \frac{|u_{ib}|}{|p_b|}$ for all $a \in A_+(i), b \in A_-(i)$. Writing $\frac{1}{\lambda_i}$ for this common ratio, we have

$$u_i \cdot z_i = \frac{1}{\lambda_i} (p \cdot z_i) = 0$$

and we conclude $U = 0$. As $U \in \Phi^{eff}$ the intersection $\Phi \cap \Gamma$ contains nothing more.

References

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Appendix: a monotonic sequence of examples

We have $N = \{1, 2\}$, $A = \{a, b, c\}$ and

	a	b	c
u_1	-1	-3	λ
u_2	-2	-1	λ

and λ takes all integer values from 4 to -3 . The first two problems, for $\lambda = 4, 3$ are positive; for $\lambda = 2$ the problem is null, then negative from $\lambda = 1$ to -3 .

